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Quantum Nonlocality From Synchronized Chaos

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Quantum entanglement suggests a deep synchronism in the physical world. The phenomenon is compared to the synchronization of a pair of chaotic systems that are loosely coupled (through only one of many degrees of freedom), so as to make inferences about the possible form of a deterministic quantum theory. A Bell's inequality is constructed for a pair of synchronously coupled variable-order Generalized Rossler Systems (GRS), with arbitrarily binarized final states. The inequality is weakly violated, despite the fact that system parameters cannot be inferred from the coupling signal. Stronger violations, more closely resembling the quantum case, are to be expected if the systems are *generally synchronized* with a non-identical correspondence function that would give anticorrelation between the binarized states. The temporal behavior of the high-dimensional chaotic systems that synchronize seems unrealistic unless the variables are only spatially asymptotic descriptions of fast processes in a space–time with micro-scale black holes or wormholes.

KEY WORDS: chaos synchronization; Bell's theorem; entanglement; EPR; space–time foam; granularity; wormholes; holographic principle; synchronicity; uncomputability; determinism; nonlocality.

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1. INTRODUCTION

The Einstein–Podolsky–Rosen (EPR) phenomenon, with the conclusions that follow from Bell's theorem, suggests a deep synchronism in the physical world. That particles in an EPR pair exist in correlated states seems to transcend, rather than to contradict, notions of an underlying causal order. If one chooses to infer nonlocality from the EPR correlations, in a noncontextual theory (e.g., Bohm, 1952), then one must explain why the nonlocal causation cannot be used to send supraluminal signals. On the other hand, if one chooses a contextual, local interpretation (e.g., Bohm and Hiley, 1981), in which the correlations are attributed

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to cosmic constraints on the observer's choice of measurement direction, then one posits a network of such relationships that far exceeds the predictions of existing physical theory. In either kind of interpretation, one seeks to explain how spatially separated physical processes might synchronize without exchanging information in any usual way.

Synchronization of regular oscillators is surprisingly common in nature. Weak signals between fireflies, between pendulum clocks on a common wall (Huygens, 1673), or between hormonal systems of cohabiting women, are known to be effective in synchronizing the respective periodic oscillators. The scope of the synchronization phenomenon was expanded greatly by the discovery that irregular chaotic oscillators could also be made to synchronize, when coupled through only a few of many dynamical variables. In this scenario, each oscillator may be effectively unpredictable, as in the quantum case, but there is a predictable relationship between oscillators. It has been suggested previously (Duane, 2001) that chaos synchronization could form the basis of apparent quantum nonlocality, if the unscrutability of the connecting "signal" in very high-dimensional systems is taken to be sufficient to preclude supraluminal transmission of information and violations of Lorentz invariance. With Bell (1964), we still take the weak nonlocality implied by the violation of his inequality to be real.

It seems unlikely that chaos alone would be sufficient to explain quantum unpredictability in a theory that is ultimately deterministic. Attempts to construct models with the requisite properties encounter obstacles such as the "fat fractal" property of basins of attraction (Ott and Sommerer, 1994), that would give increasing predictability on smaller and smaller scales, an unrealistic feature. Nevertheless, it is suggested that the synchronization phenomenon transcends the framework of deterministic chaos, at least in the usual sense of chaos that is historically based on ODE paradigms. In this view, the synchronization phenomenon will extend to deterministic systems that are even more wildly behaved. By attempting to reproduce EPR-like phenomena using synchronized chaotic systems, one can obtain clues as to the possible form of a deterministic, but highly unpredictable substructure dynamics underlying the apparent indeterminacy of quantum theory, a determinism that may be necessary after all ('t Hooft, 1999), despite the early efforts of the Copenhagen school to avoid the issue.

Synchronized chaos is described in Section 2, with attention to the phenomenon of generalized synchronization between qualitatively different systems. In Section 3, we review previous work showing that a Bell's inequality can be violated for synchronized chaotic systems without discernible transfer of information, as for an EPR pair. In Section 4, we argue that generalized synchronization will yield a Bell's inequality violation more closely resembling the EPR violation. We also argue that the class of systems that can synchronize may be less restricted in a space–time geometry with micro-scale black holes or wormholes. Conclusions are briefly summarized in Section 5.

2. BACKGROUND: SYNCHRONIZED CHAOS

While synchronization of regular oscillators with limit cycle attractors is ubiquitous in nature (Strogatz, 2003), the synchronization of chaotic oscillators has become known only recently. The phenomenon was first brought to light by Fujisaka and Yamada (1983) and independently by Afraimovich *et al*. (1987), but extensive research on the subject in the 1990s was spurred by the seminal work of Pecora and Carroll (1990), who considered configurations such as the following combination of Lorenz systems:

$$\begin{aligned} &\dot{X} = \sigma(Y - X) && \\ &\dot{Y} = \rho X - Y - XZ && \dot{Y}_1 = \rho X - Y_1 - XZ_1 \\ &\dot{Z} = -\beta Z + XY && \dot{Z}_1 = -\beta Z_1 + XY_1 \end{aligned} \tag{1}$$

which synchronizes rapidly, slaving the Y_1, Z_1-subsystem to the master X, Y, Z-subsystem. As explained by Pecora and Carroll, synchronization occurs despite the sensitive dependence on initial conditions implied by positive Lyapunov exponents, because the *conditional* Lyapunov exponents describing the Y_1, Z_1-subsystem are negative.

Systems can also synchronize when coupled *diffusively*, as with a pair of bidirectionally coupled Rossler systems:

$$\begin{aligned} &\dot{X} = -Y - Z + \alpha(X_1 - X) && \dot{X}_1 = -Y_1 - Z_1 + \alpha(X - X_1) \\ &\dot{Y} = X + aY && \dot{Y}_1 = X_1 + aY_1 \\ &\dot{Z} = b + Z(X - c) && \dot{Z}_1 = b + Z_1(X_1 - c) \end{aligned} \tag{2}$$

where α parametrizes the coupling strength.

For a pair of coupled systems that are not identical, synchronization may still occur, but the correspondence between the states of the two systems in the synchronized regime is different from the identity. In this situation, known as *generalized synchronization*, we have two different dynamical systems

$$\dot{\mathbf{x}} = F(\mathbf{x}) \tag{3a}$$

$$\dot{\mathbf{y}} = G(\mathbf{y}) \tag{3b}$$

with $\mathbf{x} \in R^N$ and $\mathbf{y} \in R^N$. If the dynamics are modified so as to couple the systems:

$$\dot{\mathbf{x}} = \hat{F}(\mathbf{x}, \mathbf{y}) \tag{4a}$$

$$\dot{\mathbf{y}} = \hat{G}(\mathbf{y}, \mathbf{x}) \tag{4b}$$

the systems are said to be generally synchronized iff there is some locally invertible function $\Phi : R^N \to R^N$ such that $||\Phi(\mathbf{x}) - \mathbf{y}|| \to 0$ as $t \to \infty$. Identical synchronization may be transformed to generalized synchronization simply by a change of variables in one system, but not the other, i.e., a change in the description of one of the systems (Rulkov *et al.*, 1995). In this situation, the correspondence function Φ is known *a priori*. Generalized synchronization may be difficult to detect without prior knowledge of Φ.

Synchronization reduces the effective dimension of the phase space by half. A synchronously coupled pair of identical systems moves on a hyperplane within the full state space of the two systems. With generalized synchronization of non-identical systems, the hyperplane becomes a *synchronization manifold* defined by the correspondence function Φ. The N-dimensional manifold in $2N$-dimensional space is $\mathcal{M} \equiv \{(p, \Phi(p)) | p \in R^N\}$. The synchronization manifold is dynamically invariant: If $x(t)$ is a trajectory of a system such as (1) or (2), for $x \in R^{2N}$, and $x(t_1) \in \mathcal{M}$, then $x(t_2) \in \mathcal{M}$ for all $t_2 > t_1$. That is, a perfectly synchronized system remains synchronized.

It is commonly not the existence, but the stability of the synchronization manifold that distinguishes coupled systems exhibiting synchronization from those that do not (such as (2) for different values of α). N Lyapunov exponents can be defined for perturbations in the N-dimensional space that is transverse to the synchronization manifold $\mathcal{M}$. If the largest of these, $h^{\perp}_{\max}$, is negative, then motion in the synchronization manifold is stable against transverse perturbations. In that case, the coupled systems will synchronize for some range of differing initial conditions. As $h^{\perp}_{\max}$ is increased through zero, the system undergoes a *blowout bifurcation*. For small positive values of $h^{\perp}_{\max}$, on–off synchronization occurs (a special case of on–off intermittency), as illustrated in Fig.1b.

Synchronization is surprisingly easy to arrange, occurring for a wide range of coupling types. Synchronization degrades through on–off intermittency or through generalized synchronization. Figure 1 depicts the two modes of degradation. In the case of degradation by intermittency, vestiges of synchronization are discernible even far from the blowout bifurcation point (Fig. 1c) (Duane, 1997). Generalized synchronization is known to occur even when the systems are very different, as in the case of a Lorenz system diffusively coupled to a Rossler system. The two systems with attractors of different dimension are known to synchronize, but the correspondence function is not smooth (Fig. 1f) (Pecora *et al.*, 1997).

Detection of generalized synchronization between different systems, coupled unidirectionally, with an unknown correspondence function is facilitated by the *auxiliary system method* (Pyragas, 1996). An identical copy of the slave system is constructed and coupled to the master system in the same manner as the first

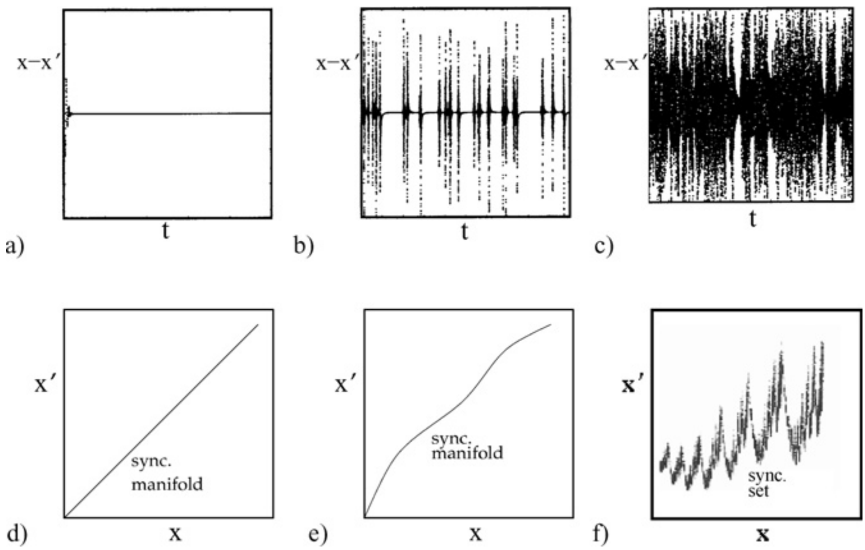

Fig. 1. Typical modes by which the synchronization between a pair of coupled systems can degrade, illustrated by the relationship between a pair of corresponding variables x and x': (a) complete, identical synchronization, (b) intermittent synchronization, (c) highly intermittent, or "partial" synchronization. Projection of the synchronization manifold onto the (x, x') plane are shown for (d) identical synchronization, (e) generalized synchronization with near-identical correspondence, (f) generalized synchronization with a correspondence function that is not smooth.

slave.

$$\dot{\mathbf{x}} = F(\mathbf{x}) \tag{5a}$$

$$\dot{\mathbf{y}}_1 = \hat{G}(\mathbf{y}_1, \mathbf{x}) \qquad \dot{\mathbf{y}}_2 = \hat{G}(\mathbf{y}_2, \mathbf{x}) \tag{5b}$$

The original master–slave system is judged to be generally synchronized if and only if the two slave systems are identically synchronized, that is, if and only if $||\mathbf{y}_1 - \mathbf{y}_2|| \to 0$ as $t \to \infty$, for any choice of initial conditions $\mathbf{x}(t = 0)$, $\mathbf{y}_1(t = 0)$, and $\mathbf{y}_2(t = 0)$. Generalized synchronization defined in this manner includes the possibility of multi-valued (i.e., multi-branched) correspondence functions (So *et al.*, 2002).

The phenomenon of chaos synchronization is not restricted to low-dimensional systems. The argument for the possible relevance of synchronized chaos to quantum nonlocality, given in the next section, is based on the fact that high-dimensional systems can synchronize when coupled through only a small number of variables. It is known, for instance, that two N-dimensional Generalized Rossler Systems (GRSs) (each equivalent to a Rossler system for $N = 3$) will synchronize for any N, no matter how large, when coupled via only one of the N

variables:

$$\begin{array}{llll} \dot{x}_1^A & = -x_2^A + \alpha x_1^A + x_1^B - x_1^A & \dot{x}_1^B & = -x_2^B + \alpha x_1^B + x_1^A - x_1^B \\ \dot{x}_i^A & = x_{i-1}^A - x_{i+1}^A & \dot{x}_i^B & = x_{i-1}^B - x_{i+1}^B \qquad i = 2 \ldots N-1 \\ \dot{x}_N^A & = \epsilon + \beta x_N^A (x_{N-1}^A - d) & \dot{x}_N^B & = \epsilon + \beta x_N^B (x_{N-1}^B - d) \end{array} \tag{6}$$

Each system has an attractor of dimension $\approx N - 1$, for N greater than about 40, and a large number of positive Lyapunov exponents that increases with N.

Early interest in synchronized chaos was based to a large degree on the promise of an application to secure communications (Cuomo and Oppenheim, 1993; Cuomo *et al.*, 1993). Security was based on the idea that a time series in the one variable coupling the two systems would be difficult to distinguish from noise, especially for high-dimensional systems such as (6) (e.g., Parlitz and Kocarev, 1997). A signal composed of such a time series might be meaningful only to a receiving system with parameters identical to those of the sending system, or at least with a known correspondence function between states of the two systems in the synchronized regime. The existence of synchronized chaos in naturally occurring systems was made more plausible by demonstrations of synchronization in spatially extended systems governed by PDEs in one space dimension (Kocarev *et al.*, 1997) and in 2D fluids (Duane and Tribbia, 2001).

3. EPR-LIKE PHENOMENA IN SYNCHRONIZED CHAOTIC SYSTEMS AND VIOLATION OF AN ANALOGUE BELL'S INEQUALITY

That EPR correlations cannot possibly result from pre-assigned spin values in an objective local theory follows from Bell's theorem, specifically from the violation of Bell's famous inequality. To pursue the analogy between synchronized chaos and quantum entanglement, we construct an analogous inequality for bidirectionally coupled, synchronized Generalized Rossler Systems (6). The GRS is chosen as a simple example of a dynamical system with tractable variable-order behavior. (Higher dimensional PDE systems are often less suitable because they have attractors of limited dimension.) No specific correspondence between GRS dynamical variables and physical quantities is implied, but the system parameters α^A and α^B are taken to be physical quantities of some sort, analogous to arbitrarily chosen measurement orientations. Each system is imagined to collapse to one of two final states at measurement time T, depending on whether $x_1(T) > 0$ or $x_1(T) \leq 0$. The dynamical variables x_i at any given time, say $t = 0$, are the "hidden variables" of the system. The configuration described by (6) is related to one with unidirectional coupling that was proposed as the basis of a secure communications scheme (Parlitz and Kocarev, 1997). If we take the two subsystems to be spatially separated, the interaction is nonlocal (in the same sense as is Newtonian gravity).

Correlations can be defined as in the quantum-mechanical case, now as functions of system parameters (say α^A and α^B) in place of the usual measurement orientations:

$$P(\alpha^A, \alpha^B) \equiv \int \rho(\Lambda) A(\alpha^A, \alpha^B, \Lambda) B(\alpha^B, \alpha^A, \Lambda) d\Lambda \tag{7}$$

where now A and B are each ± 1 depending on the state of the respective subsystem at time $t = T$, $\Lambda \equiv (x_1^A(0), x_2^A(0), \ldots x_N^A(0); x_1^B(0), x_2^B(0), \ldots x_N^B(0))$ is shorthand for the state of the entire system at $t = 0$, i.e., the hidden variables, and ρ is a distribution function for such initial states. It is required that $\int \rho(\Lambda) d\Lambda = 1$, but any function ρ satisfying this normalization requirement will suffice. A conveniently computed estimate of P is obtained by fixing the initial state Λ_o and instead varying the time T of the "collapse" to define A_T and B_T. Assuming ergodicity: $P(\alpha^A, \alpha^B) \approx \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} A_T(\alpha^A, \alpha^B, \Lambda_o) B_T(\alpha^B, \alpha^A, \Lambda_o) dT$ since varying T is equivalent to varying the time at which the initial state is defined.

It is assumed that the subsystems are synchronized or generally synchronized (i.e., there is a one–one correspondence between states that is not the identity) over the range of integration. If the final states A and B are only functions of the corresponding α parameter, i.e., $A = A(\alpha^A, \Lambda)$ and $B = B(\alpha^B, \Lambda)$, then one can establish an analogue Bell's inequality. One introduces a third subsystem with parameter $\alpha^{B'}$, that is also coupled to a subsystem identical to the A subsystem, and considers the difference

$$\begin{aligned} P(\alpha^A, \alpha^B) - P(\alpha^A, \alpha^{B'}) &= \int d\Lambda \rho(\Lambda) [A(\alpha^A, \Lambda) A(\alpha^B, \Lambda) \\ &\quad - A(\alpha^A, \Lambda) A(\alpha^{B'}, \Lambda)] \\ &= \int d\Lambda \rho(\Lambda) A(\alpha^A, \Lambda) A(\alpha^B, \Lambda) \\ &\quad \times [-A(\alpha^B, \Lambda) A(\alpha^{B'}, \Lambda) + 1] \end{aligned} \tag{8}$$

having used $A = \pm 1$, $B = \pm 1$, and having substituted $B(\alpha^B, \Lambda) = A(\alpha^B, \Lambda)$. It follows from (8) that

$$|P(\alpha^A, \alpha^B) - P(\alpha^A, \alpha^{B'})| \leq \int d\Lambda \rho(\Lambda) [1 - A(\alpha^B, \Lambda) A(\alpha^{B'}, \Lambda)] \tag{9}$$

which is Bell's inequality:

$$|P(\alpha^A, \alpha^B) - P(\alpha^A, \alpha^{B'})| \leq 1 - P(\alpha^B, \alpha^{B'}) \tag{10}$$

Noting that $P(\alpha^B, \alpha^B) = 1$, one sees that the left hand side of (10) is in general $O(\alpha^B - \alpha^{B'})$ whereas the right hand side of (10) is in general $O[(\alpha^B - \alpha^{B'})^2]$ if P is a smooth function, a contradiction. The inequality (10) is analogous to Bell's inequality in quantum theory, which differs from (10) only in the sign of

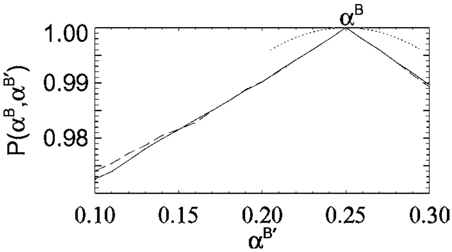

Fig. 2. The Bell correlation $P(\alpha^B, \alpha^{B'})$ between the final binarized states of the coupled GRSs (6), for varying parameter $\alpha^{B'}$ and fixed $\alpha^B = 0.25$. Correlations are shown for a GRS of order $N = 11$ (*solid line*) and $N = 101$ (*dashed line*). The discontinuity in first derivative at $\alpha^B = \alpha^{B'}$ required by Bell's inequality is evident in both cases. The dotted line suggests the form of the correlations for a pair of coupled systems that would resemble a quantum-mechanical EPR pair. Correlations were computed by numerical integration, using a Runge–Kutta algorithm with adaptive stepsize control, over intervals $T_2 - T_1 = 10^5$, with the other GRS parameters: $\epsilon = 0.1$, $b = 4$, $d = 2$. The average computational error in the correlation values is 2×10^{-4}.

the last term on the right hand side. The difference arises because spins in an EPR pair are anticorrelated ($P(\alpha, \alpha) = -1$) instead of correlated, as with the GRS pair. (In standard quantum theory $P(a, b)$ is indeed the smooth function $-\cos(a - b)$, where a and b are measurement orientation angles.)

The analogue Bell's inequality (10) might be expected to fail since the configuration of GRSs includes an explicit coupling signal. However, a naive observer, unable to extract the system parameters from the signal or to distinguish the signal from background noise as $N \to \infty$, would expect the inequality to hold. For a configuration in which the signal is supraluminal, this observer would detect no violation of causality and would claim that causality required that the analogue Bell's inequality be satisfied.

The correlations plotted in Fig. 2 exhibit the discontinuity in first derivative at identical parameter values $\alpha^B = \alpha^{B'}$ which is characteristic of theories that do satisfy Bell, unlike quantum systems. However, the full inequality (10), involving all three parameter values, is violated when α^A is far from $\alpha^B = \alpha^{B'}$. The differences on the two sides of the inequality (10) both estimate derivatives, which are plotted in Fig. 3 showing violations of the inequality for large parameter differences. These violations imply that the states of the two systems, cannot, in

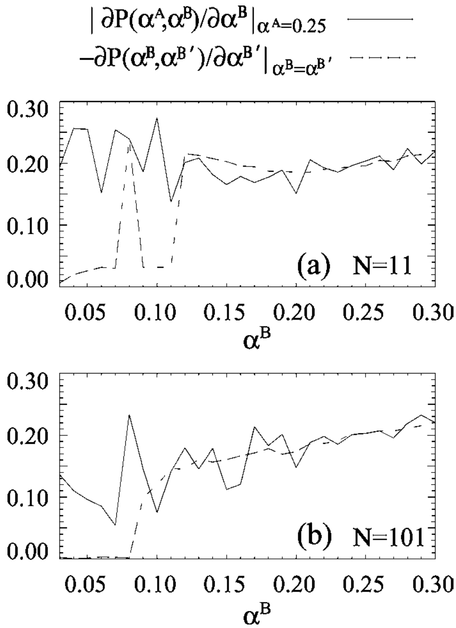

Fig. 3. The LHS (*solid line*) and RHS (*dashed line*) of Bell's inequality (10), after division by $\alpha^B - \alpha^{B'}$, for $\alpha^B - \alpha^{B'} = .01$, for coupled GRSs of order (a) $N = 11$ and (b) $N = 101$. The quantities plotted for varying α^B and fixed $\alpha^A = 0.25$ estimate the partial correlation derivatives $|\partial P(\alpha^A, \alpha^B)/\partial\alpha^B|_{\alpha^A=.25}$ (*solid line*) and $\partial P(\alpha^B, \alpha^{B'})/\partial\alpha^{B'}|_{\alpha^{B'}=\alpha^B}$ (*dashed line*). Violations of Bell's inequality are evident in both panels for α^B much smaller than $\alpha^A = 0.25$. Correlations were computed as in Fig. 2, except with a longer integration time $T_2 - T_1 = 10^6$ for the dashed line in (a) only. The average computational error is 0.05 for the solid line in (a), 0.002 for the dashed line in (a), 0.05 for the solid line in (b), and 0.006 for the dashed line in (b).

general, be viewed as the result of joint initial conditions followed by separate evolution—the analogue of the quantum EPR paradox.

That Bell's inequality is satisfied for $\alpha^A \approx \alpha^B$, as seen in Fig. 3, is not surprising, since the coupling term linking the two systems, $x_1^A - x_1^B$, vanishes in the case of identical synchronization with $\alpha^A = \alpha^B$. Unlike the quantum case, the two systems can be regarded as separable when their parameter values are identical. An explanation of the discontinuity in the first derivative of correlation, seen in Fig. 2, also requires that Bell's inequality be satisfied in a neighborhood of $\alpha^A = \alpha^B$, a circumstance that can be understood heuristically in terms of the typical behavior of generally synchronized systems. When the parameters of the two systems differ slightly, a synchronization manifold is defined by a smooth invertible mapping, here $\phi : \{x_i^A\} \to \{x_i^B\}$, between the states of the two systems. For such small differences, the mapping is typically given by a change of dynamical variables in one system (Rulkov *et al.*, 1995), which is in fact defined on the synchronization manifold by ϕ itself. The two synchronized systems exhibit precisely the same dynamics, described differently. With both systems belonging to a family of systems that differ among themselves only by changes of variables, let $\hat{\Lambda} \equiv \{\hat{x}_i^A(0), \hat{x}_i^B(0)\}$ denote the initial state in some canonical set of dynamical variables. Then, the final binarized state of each system depends only on the particular binarization that defines A and B in terms of the $\hat{x}_i$, so we can write:

$$A = A(\alpha^A, \hat{\Lambda}) \qquad\qquad B = B(\alpha^B, \hat{\Lambda}) \tag{11}$$

assume a fixed $\rho(\hat{\Lambda})$, and prove the analogue Bell's inequality (10). That the inequality (10) is satisfied even for a restricted class of parameter values $\alpha^A \approx \alpha^B \approx \alpha^{B'}$ still leads to a contradiction, as in the discussion following (10), unless there is a discontinuity in the first derivative of correlation at identical parameter values, which is seen in Fig. 2.

When the parameters differ widely, one can no longer define $\hat{\Lambda}$ and write (11). One must consider $A = A(\alpha^A, \alpha^B, \Lambda)$, $B = B(\alpha^A, \alpha^B, \Lambda)$, but a more basic issue is that Λ—which specifies the initial *paired* states—cannot be varied independently of the dynamical parameters α^A and α^B, since the correspondence between states in generalized synchronization varies with these parameters. In general, one must write:

$$\rho = \rho(\Lambda, \alpha^A, \alpha^B) \tag{12}$$

blocking the steps analogous to (8)–(9) in the proof of Bell's inequality, which depend on an independent $\rho = \rho(\Lambda)$. The dependance of ρ on the dynamical parameters is greater when the two systems begin to exhibit qualitatively different dynamical behavior. Meyer *et al.* (1997) reported a change in dynamical behavior around the value $\alpha = .07$ where violations of Bell's inequality are seen (Fig. 3). The change is evident in Fig. 4 showing that the dimension of the attractor for $\alpha = .07$, for any order N, is lower than the dimension for the higher values of α. Nevertheless, generalized synchronization between directionally coupled GRSs with parameter values $\alpha^A = .07$ and $\alpha^B = .25$ was verified using the auxiliary system method described in Section 2. It was inferred that the corresponding bidirectionally coupled pair also synchronizes.

The violations of Bell's inequality seen in Fig. 3 appear to persist as the order N of the GRSs is increased. If the violation is quantified as the deviation from unity of the ratio of the quantities represented by the solid and dashed lines, the

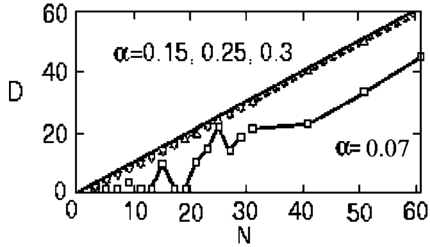

Fig. 4. Lyapunov dimension D of the attractor of a GRS of order N for varying N and different values of the parameter α; $\alpha = 0.3$ (∇), $\alpha = 0.25$ (△), $\alpha = 0.15$ (◯), $\alpha = 0.07$ (□). Other parameters are as in Fig. 2. (re-created from Meyer *et al.*, 1997).

violation is seen to remain constant or to increase with order N. The argument given above for the origin of the violations should apply regardless of order, since Fig. 4 implies that the qualitative system dynamics change with shifting α in a like manner for all N. However, as suggested by Takens' theorem, the dynamical parameters are increasingly well masked in the coupling signals (consisting of the variables x_1^A,x_1^B) as N increases, for all parameter settings. Parlitz and Kocarev (1997) explicitly showed that as N increases, one must consider an increasing length of signal to distinguish the coupling signal from noise with the same power spectrum. Thus, in the limit $N \to \infty$, it is expected that violations of Bell's inequality remain, these violations regarded as a property of the formal system (6), but the signals linking the subsystems are indistinguishable from noise.

4. IMPLICATIONS FOR THE FORM OF A DETERMINISTIC QUANTUM THEORY

It is the principal thesis of this work that the universe is perpetually in a state of generalized synchronization among its parts. In the usual situation, the correspondence function that defines the synchronization is completely intractable, and the connections that maintain the synchronization appear meaningless. However, in a sufficiently symmetrical situation, such as that of an EPR pair, the correspondence is simple.

The hidden relationships in the usual situation are one way of realizing Bohm's "implicate order" (Bohm, 1980). A second historical source for the proposed use of synchronization is the Jung–Pauli concept of "synchronicity" (Jung and Pauli, 1955). Jung and Pauli envisioned an "acausal connecting principle," existing alongside causality, as suggested by Fig. 5. To Pauli it was important that synchronicities were meaningful, isolated events. Unfortunately, he kept his speculations about synchronicity separate from his scientific work and expressed

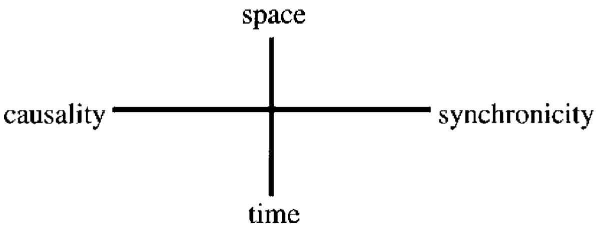

Fig. 5. Diagram constructed by Carl Jung, later modified by Wolfgang Pauli, to suggest relationships based on synchronicity as an "acausal connecting principle," existing alongside causal relationships.

a concern that they might require science to abandon its attempt to describe nature rationally. But if we allow bursting away from the synchonization manifold, as in Fig. 1, so that synchronization obtains only in an average sense, then perhaps a rational explanation for isolated synchronicities can be provided.

What then can be learned from the phenomenon of synchronization of loosely coupled chaotic systems? The most obvious question is how to obtain violations of Bell's inequality that are qualitatively stronger, more like the quantum case. If this question is linked to the question of how to get anticorrelated systems, then the general form of the answer to both questions is apparent. In the GRS study, it was seen that Bell's inequality was not violated when the parameters of the two systems are the same, since in that case the coupling signal vanishes, and the systems evolve (assuming they are completely isolated) as would uncoupled systems that are initially synchronized. With anticorrelating systems, on the other hand, the coupling signal analogous to $x_1^B - x_1^A$ does not generally vanish at the point where parameters are equal and P is extremal, since the states of the two systems are not the same there. Whatever dynamics gives a correspondence function that results in anticorrelation between two systems with identical parameters cannot be equivalent to the vanishing of the coupling signal. Of course, the dynamics must have realistic symmetry properties under rotation and particle exchange. But provided that the symmetry requirements can be satisfied and that generalized synchronization gives anticorrelated states, stronger violations of Bell's inequality are to be expected.

A greater challenge is to find high-dimensional systems that are sufficiently realistic and still synchronize when loosely coupled. The GRS has the properties that its "metric entropy," the sum of the positive Lyapunov exponents, $\sum_{h_i>0} h_i$, is constant as $N \to \infty$, and that its largest Lyapunov exponent $h_{\max} \to 0$ as $N \to \infty$. In other words, the higher the dimension, the less chaotic the system. Such behavior is suspect in a system intended to represent unpredictable quantum fluctuations. It is not known whether systems that are more chaotic than the GRS, but with attractors of arbitrarily high dimension, can be made to synchronize with loose coupling.

From one vantage point, however, the GRS behavior may be physically reasonable. Perhaps the GRS should be viewed as a spatially asymptotic description of an intrinsically faster dynamics in a highly curved space–time. That is, if the physical system that the GRS describes lives in the vicinity of a micro-black hole or wormhole, the variables in the asymptotic description will be slowed, but the actual physical processes will be realistically violent. The idea of Planck-scale granularity in space–time is not new (e.g., Bombelli *et al.*, 1987; Hawking, 1978) and is consistent with recent experimental evidence (Amelino-Camelia and Piran, 2001). Foam has usually been conceived as arising from the quantization of classical general relativity, but our scenario presupposes a foam-like structure at a classical deterministic level, a structure that might arise from a generally covariant but scale-dependent modification of Einstein's equations to generate non-trivial micro-scale behavior. Basing a deterministic quantum theory on such geometry might allow uncomputability to enter, a desideratum that was put forth by Penrose (1991), since the question of whether two foam geometries are topologically equivalent is uncomputable.

The actual deterministic quantum theory should be expressible as a PDE on space–time, on all but the smallest scales. Interestingly, synchronization has been found in coupled pairs of 1D systems and 2D systems (i.e., two space dimensions) but not thus far in 3D systems. It has been conjectured that for turbulent fluid systems, the 2D synchronization results would not carry over to 3D because of the different direction of energy cascade, from large to small scales, in 3D turbulence (Duane and Tribbia, 2001). (The turbulent cascade was recently used to motivate a deterministic quantum theory in another way by Palmer (2004).) If the PDE system for the deterministic quantum theory must also be 2D to allow synchronization, this constraint would be consistent with the *holographic principle* according to which nature is fundamentally 2D on the Planck scale ('t Hooft 1993). It is tempting to think of two such 2D PDE systems permanently synchronized by virtue of a sub-Planck scale wormhole connecting them that is not traversable by matter because of its width, with a GRS-like representation of each system at large distances, as in Fig. 6. (Fast variables describing processes away from the hole could also be added.) However, the wormhole may be no more than a metaphor for the nonlocal sub-Planck scale physics.

Finally, we note that the individual particles in our scheme will have to be rather complex, multi-parameter entities, in order for the signal from one member of an EPR pair to be meaningful only to its partner. While multi-parameter descriptions of individual particles would seem to conflict with field-theoretic descriptions, the idea is consistent with older ideas as expressed by Bohm and Hiley (1993).

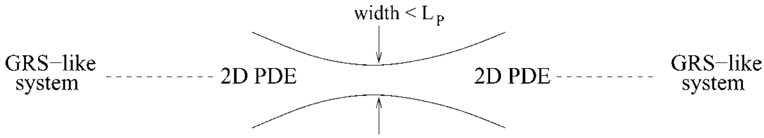

Fig. 6. Nonlocal connections as mediated by "wormholes," of width less than the Planck length L_P, in a granular space–time. The "wormholes" are not traversable by material particles but enable synchronization of spatially separated systems, with 2D dynamics at the Planck scale, that are represented asymptotically by synchronizing GRS-like systems.

5. CONCLUSIONS

In summary, synchronism is a likely feature of any deterministic theory of quantum phenomena. Since chaos synchronization is a well-researched phenomenon, it can shed light on the necessary form of such a theory. The temporal behavior of high-dimensional synchronizing systems, for which the nonlocal coupling is disguised, appears to be unrealistic in a flat space–time. Thus, small-scale space–time structure seems to be required. We have noted that this conclusion is consistent with other evidence that space–time is granular on such scales.

We have also supported the very notion of a deterministic quantum theory, in a way that seems to agree with Pauli's idea of synchronicity, Bohm's "implicate order," Penrose's requirement that a deterministic quantum theory be uncomputable, and 't Hooft's ideas regarding determinism and holography. The next step is to further consider foam-like space–time structures, introduced at a classical level, that can be used in conjunction with synchronization.

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